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STEADY VISCOUS FLUID FLOW IN PLANE CHANNELS FORMED BY SECTIONS OF COAXIAL CIRCULAR CYLINDER†

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An exact solution of the plane problem for the Navier–Stokes equations is obtained when there is radial symmetry for the trajectories of the fluid particles. Unlike existing solutions the pressure gradient in the angular direction is taken into account. Copyright © 1996 Elsevier Science Ltd.

1. The following equations hold for plane steady fluid motion over concentric circles

$$u_r = 0, \quad u_\theta = u(r) \tag{1.1}$$

where r and θ are polar coordinates, and u_r and u_θ are the components of the velocity in the direction of the radial and angular coordinates, respectively. In addition to conditions (1.1) the following conditions for the pressure is also usually specified in advance [1–3]

$$p = p(r) \tag{1.2}$$

Taking (1.1) into account this condition leads to a second-order ordinary differential equation for the function $u(r)$, obtained from the Navier–Stokes equations. Integration of this equation gives

$$u = Mr + N/r \tag{1.3}$$

where M and N are arbitrary constants.

The function (1.3) is usually employed [1–3] to solve the problem of the steady fluid motion between two rotating coaxial cylinders.

We will also assume below that the trajectories of the fluid particles have radial symmetry, but we will not use condition (1.2) for the pressure here. The density and coefficient of viscosity of the fluid will be assumed to be constant.

We will consider the equation for the stream function

$$\frac{\partial\psi}{\partial x_2}\frac{\partial\Delta\psi}{\partial x_1} - \frac{\partial\psi}{\partial x_1}\frac{\partial\Delta\psi}{\partial x_2} = \nu\Delta\Delta\psi, \quad \Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \tag{1.4}$$

in rectangular coordinates, which is obtained after eliminating the pressure from the Navier–Stokes equations for plane motion.

The solution of Eq. (1.4) will be sought in the class of functions $\psi = \psi(r)$, $r = \sqrt{(x_1^2 + x_2^2)}$. It can be shown that in this case the left-hand side of Eq. (1.4) is identically zero, which leads to a biharmonic equation. Using the Goursat representation of the biharmonic function ψ in terms of the harmonic functions ψ_1 and ψ_2, $\psi = r^2\psi_1 + \psi_2$ [4], we obtain the solution

$$\psi = r^2(A + B\ln r) + D + C\ln r \tag{1.5}$$

where A, B, C and D are arbitrary constants.

The projections of the velocity are

$$u_\theta = -\frac{\partial\psi}{\partial r} = -(2A + B)r - 2Br\ln r - \frac{C}{r}, \quad u_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} \equiv 0 \tag{1.6}$$

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Solution (1.6) was obtained previously in [2] directly from the Navier–Stokes equations. We will establish a relationship between this solution and the accurate solution of Hamel's equation (1.4)

$$\psi = f(\varphi), \quad \varphi = -\frac{2(a\ln r + b\theta)}{a^2+b^2}, \quad a,b = \text{const} \tag{1.7}$$

When $b = 0$ the function f satisfies the equation

$$f''' + 2af'' + a^2 f' = c, \quad c = \text{const} \tag{1.8}$$

Using the relation $f' = aru_\theta/2$, we can write instead of (1.8)

$$\frac{d^2u_\theta}{dr^2} - \frac{1}{r}\frac{du_\theta}{dr} + \frac{1}{r^2}u_\theta = \frac{8}{a^3r^3}c \tag{1.9}$$

It can be shown by direct substitution that the function (1.3) satisfies Eq. (1.9) when $c = Na^3/2$.

It turns out that the function $u_\theta(r)$, defined by the first relation of (1.6), satisfies Eq. (1.9); this can be shown by direct substitution.

Taking (1.6) into account we can write the Navier–Stokes equations in polar coordinates

$$\frac{1}{\rho}\frac{\partial p}{\partial r} = \frac{1}{r}\left[(2A+B)r + 2Br\ln r + \frac{C}{r}\right]^2, \quad \frac{1}{\rho}\frac{\partial p}{\partial \theta} = -4B\nu$$

Integrating this system we obtain

$$\frac{p}{\rho} = (2A+B)^2\frac{r^2}{2} - \frac{C^2}{2r^2} + 2C(2A+B)\ln r + 2B^2r^2\left(\ln^2 r - \ln r + \frac{1}{2}\right) +$$

$$+2B(2A+B)r^2(\ln r - \tfrac{1}{2}) + 4BC\ln|\ln r| - 4B\nu\theta + \text{const} \tag{1.10}$$

The shear friction stress between the ring layers is

$$\tau = \mu\left(\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right) = \mu\left(-2B + \frac{2C}{r^2}\right) \tag{1.11}$$

where μ is the dynamic coefficient of viscosity.

It can be seen by comparing (1.3) and (1.6) that solutions of the form (1.3) can be obtained from (1.6) if we put $B = 0$.

Note that although the trajectories of the fluid particles have radial symmetry ($\psi = \psi(r)$), the pressure, as can be seen from (1.10), is a function not only of r but also of θ when $B \neq 0$.

The presence of three arbitrary constants in (1.6), unlike (1.3), enables boundary-value problems with adhesion conditions on fixed walls to be solved.

2. Solution (1.10) obviously only makes sense when $\theta \in (0, 2\pi)$, i.e. in a plane channel formed by sections of coaxial cylinders of radii R_1 and R_2 ($R_1 \leq r \leq R_2$, $0 < \alpha \leq \theta \leq \beta < 2\pi$. The adhesion conditions: $u_\theta = 0$ when $r = R_1$ and $r = R_2$ must be satisfied on the channel walls. We will assume that we know the fluid flow rate Q through the channel formed by sections of coaxial cylinders whose heights in the direction of the z axis are equal to unity and whose cross-section forms the plane channel considered (this flow rate is equal to $\psi(R_2) - \psi(R_1)$). These three conditions enable a system of equations to be obtained for determining the three arbitrary constants that occur in (1.6)

$$2R_iA + (R_i + 2R_i\ln R_i)B + R_i^{-1}C = 0, \quad i = 1,2 \tag{2.1}$$

$$(R_2^2 - R_1^2)A + (R_2^2\ln R_2 - R_1^2\ln R_1)B + (\ln R_2 - \ln R_1)C = Q$$

The determinant of system (2.1)

$$\Delta = 2R_1R_2\left[2\ln^2 k - \frac{1}{2k^2}(k^2-1)^2\right], \quad k = \frac{R_2}{R_1}$$

is negative when $k > 1$ in view of the inequality $2\ln k < k - 1/k$, which is obtained by integrating the inequality $2/k < 1 + 1/k^2$ in the limits from $k = 1$ to k. Hence, system (2.1) has the unique solution

$$A = \Delta_1 / \Delta, \quad B = \Delta_2 / \Delta, \quad C = \Delta_3 / \Delta \qquad \Delta_1 = Q\left\{\frac{1-k^2}{k} - \frac{2}{k}[(k^2-1)\ln R_1 + k^2 \ln k]\right\}, \qquad \Delta_2 = 2Q(k^2-1)/k, \quad \Delta_3 = 4QR_1^2 k \ln k \tag{2.2}$$

Substituting (2.2) into (1.6), (1.10) and (1.11) we obtain relations for determining the corresponding flow parameters in the channel considered.

Note that an approximate solution of a similar problem was obtained previously in [2] assuming the pressure to be constant over the radius, when $p = p(\theta)$.

3. For convenience, we will consider the relations obtained in dimensionless form, introducing scales of velocity V and length R_1. The quantity u_θ in (1.6) must then be replaced by u_θ/V, Q must be replaced by Q/VR_1, R_1 must be replaced by 1, R_2 must be replaced by $R_2/R_1 = k = 1 + h$, and r must be replaced by $r/R_1 = 1+y$.

Assuming the dimensionless width of the gap h between the cylinders (and, of course, y also), to be a small quantity, we expand the quantities containing $r = 1+y$ in (1.6) in series and confine ourselves to terms $O(y^2)$ inclusive. We thereby obtain the approximate expression

$$u = a + by + my^2 \tag{3.1}$$

$$a = -(2A + B) - C, \quad b = -2B + 2C + a, \quad m = -B - C$$

The first relation of (2.1) with $i = 1$ gives $a = 0$. Expanding $\ln k = \ln(1 + h)$ in series and confining ourselves in the numerators and denominators to terms of the lowest order of smallness, we can determine b and m. Substituting these quantities into (3.1) we obtain

$$u = 6Qh^{-3}(-hy + y^2) \tag{3.2}$$

This expression is the solution of the problem of the rectilinear motion of a viscous fluid between two parallel fixed walls, spaced a distance h from one another (plane Poiseuille flow).

It can be regarded as an approximate relation for the fluid velocity in the channel formed by sections of coaxial circular cylinders, the gap between which is small compared with the mean radius of curvature of the channel.

Figure 1 shows some results of calculations. We have assumed $Q = 2/3$. Curves 1 and 2 were obtained from (1.6) and (2.2), and the parabola 3 corresponds to formula (3.2). Comparing these, we must bear in mind that for the first and third curves $h = 10$ (correspondingly, $\delta = 10$), and for the second and third curves $h = 1$ ($\delta = 1$). For $h = 0.01$ ($\delta = 0.01$) the curves practically coincide.

4. We have so far considered motions in a channel with fixed walls. We will now consider a rotating cylinder surrounded by part of a circular cylinder coaxial with it. The boundary conditions will then be

$$u_\theta = U \quad \text{for} \quad r = R_1, \quad u_\theta = 0 \quad \text{for} \quad r = R_2$$

(we have reverted to dimensional quantities). In this case we must put U instead of zero on the right-hand side of

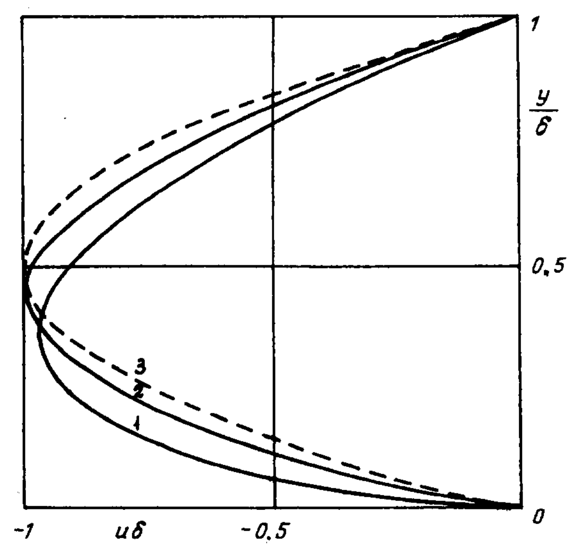

Fig. 1.

Eq. (2.1) when $i = 1$. Taking this into account we obtain

$$A = \frac{\Delta_1 + \Delta_4}{\Delta}, \quad B = \frac{\Delta_2 + \Delta_5}{\Delta}, \quad C = \frac{\Delta_3 + \Delta_6}{\Delta}$$

$$\Delta_4 = -UR_2\left[\left(2\ln k - 1 + \frac{1}{k^2}\right)\ln R_1 + 2\ln^2 k\right]$$

$$\Delta_5 = UR_2\left(2\ln k - 1 + \frac{1}{k^2}\right), \quad \Delta_6 = -UR_1^2 R_2(2\ln k + 1 - k^2)$$

In the limit of a small relative gap the term $U(1 - 4y/h + 3y^2/h^2)$ is added to the right-hand side of (3.2). This is the solution of the problem of Couette flow.

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